Visualizing Spacetime

When we want to find something, we need to know not only where it is, but also when it's there. Thus in addition to the three numbers needed to locate position in our three-dimensional world, we need a fourth number--time. We thus need four coordinates to locate something in our world. This gives us the clever idea of treating time as a direction that gives us a fourth dimension! Let's see if this works.

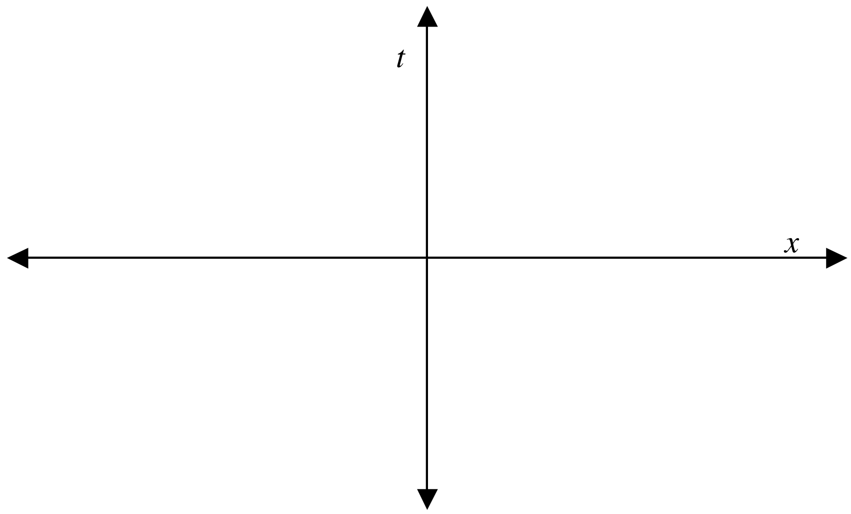
Think of yourself as moving through time, not worrying about whether or not this means anything. Stand still, close your eyes, and feel the motion through time. Are you moving in space? Not if you obeyed my instruction to stand still! So if time is thought of as a direction, it is a direction that you can move along without moving in any of the three spatial directions! It thus defines a fourth dimension! If all this seems too simple to you, you are catching on. It is simple, once we are clear about what we mean by dimension!

Now does this mean anything? What can we mean by saying that we are moving through time? Motion is change in position. If we think of time as a fourth coordinate, then the value of this coordinate certainly changes--in fact we cannot stop it from changing! We cannot stop time, though many have tried! In this sense we are certainly always moving through time. The time coordinate always changes. In this class we are going to combine the observation that we move through time with the observation that the direction we move in time seems to define a fourth dimension. We are going to find that taking this viewpoint leads us to a picture of reality that makes the results of relativity theory very natural. This, to me, says that this idea of motion through time means quite a lot.

Enough philosophy. What we have arrived at is that the space we live in is actually four-dimensional. This does not mean that all four dimensions are alike. There is something very special about the time dimension. This is reflected in the fact that you cannot stop moving through time, though you can stand still with respect to the three other (spatial) dimensions. We will never come to understand why time is different (no one does), but we will, by carefully examining this difference, understand many deep facts about our world.

We start by trying to visualize our four dimensional world. There is immediately a problem. This world is four-dimensional and we only know how to visualize in (at most) three dimensions. What to do? We will have to somehow reduce the number of dimensions that we are trying to visualize!

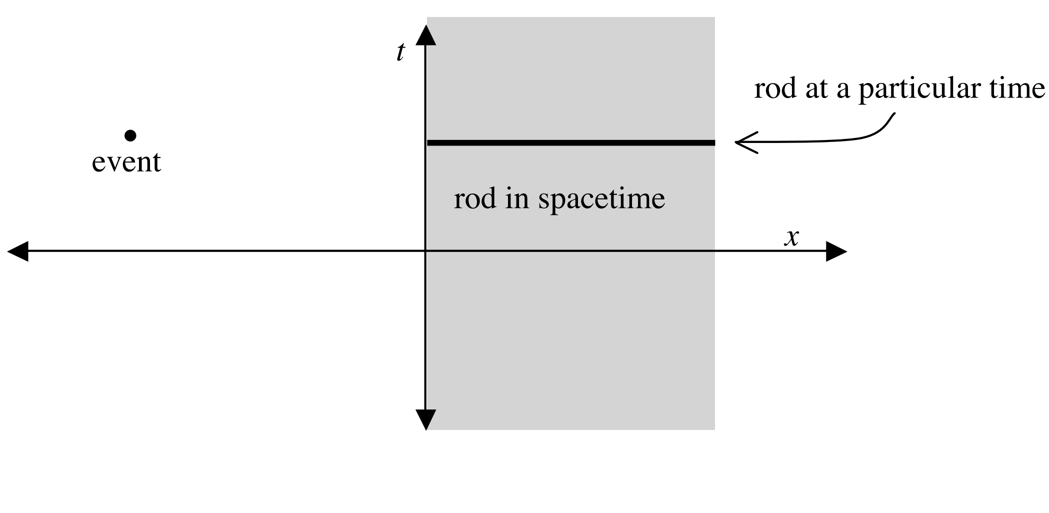
The normal space part of spacetime is three-dimensional. It is a fortunate fact that the three spatial directions are completely equivalent. If we understand one, we understand the others. We can thus forget one of the spatial directions (let's drop the z direction), and reduce our problem to trying to visualize a three dimensional spacetime, with a time direction and two spatial directions. We have succeeded in making the world possible to visualize. However, not everybody finds it easy to visualize things in three dimensions. But wait! We still have a two dimensional spatial part, and these two spatial directions are equivalent. For your sake and for the sake of my drawings let's drop another spatial direction (the y direction). We are now left with a two dimensional spacetime with the time direction, denoted by t, and the *x* direction:



This is a ***spacetime diagram***. We will be seeing a lot of this.

This diagram allows us to see motion in the *x* and t directions only. This is the same as saying that I will only move sideways (in the *x* direction) and, unavoidably, in the time direction. What is a point in our spacetime diagram? It is something with two coordinates, a value for *x* and a value for t. So it is something that happened in a particular *place* at a particular *time*. This is called an ***event***. Events are the points on a spacetime diagram. Examples of events are: a bat hitting a ball, a hand clap, or a firecracker going off. Note that these examples are not exactly events, as they take a (very short) amount of time to occur, but they are the closest that everyday things come.

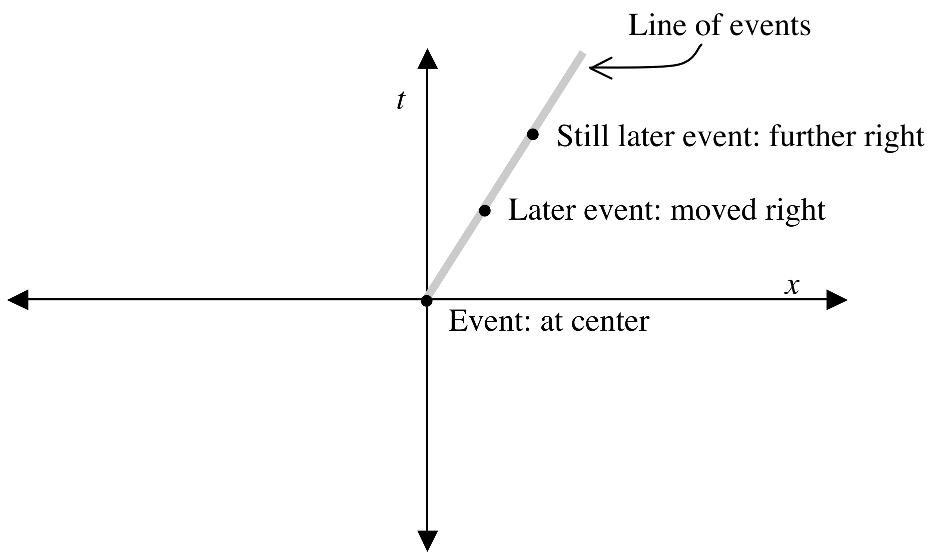
What does an object in the real world look like on our spacetime diagram? Since we only have one spatial dimension, think about a very thin rod. Idealize this rod as a one-dimensional object. Put the end of the rod at the center of the diagram. As time goes by, the rod will sweep out an area on the spacetime diagram, for it is not moving in space but is moving in time. Thus an object in the real world looks like a collection of events on our spacetime diagram:



The rod, which we idealize as a one-dimensional object in space is (unavoidably) a two dimensional object in spacetime. You can think of this in several ways. I think of the rod as 'sweeping out' a two dimensional object in spacetime. It is perhaps more accurate to think of the rod as fundamentally two dimensional and we experience seeing one dimensional 'slices' of the rod at any one time (see the above diagram).

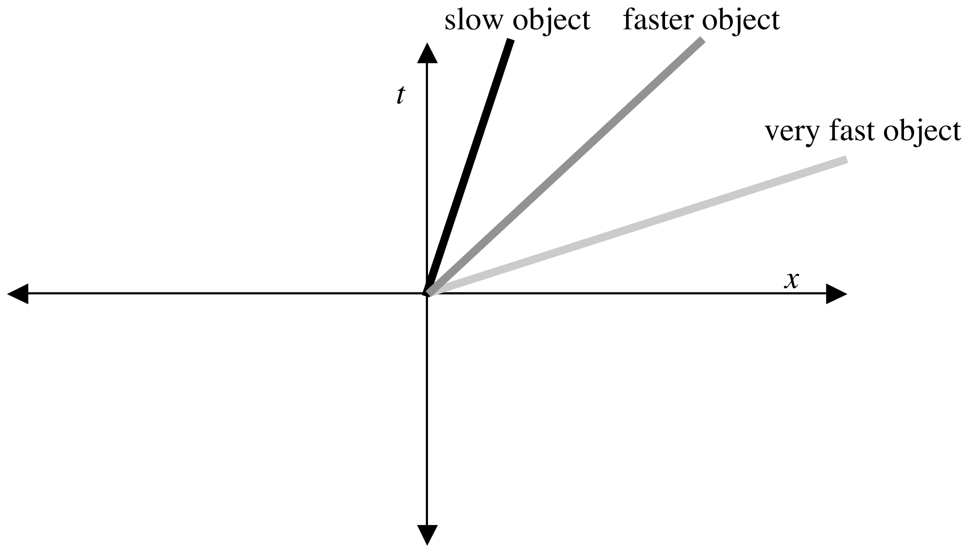
For the rest of these notes, I am only going to think about objects that are like points in our three dimensional space. Though we say that these points are zero dimensional objects in three-space, like the rod they 'sweep out' one-dimensional lines on our spacetime diagram. Try to visualize the appropriate spacetime diagram before you go on.

We can now think about how motion of our point looks on a spacetime diagram. Let's imagine our point starting at the center (so that the x coordinate and time coordinate are zero) and moving steadily to the right. After a certain time, say three seconds, the point will be off to the right. This is an event in the upper right section of the diagram. After a later time we will have another event with the point further to the upper right. Further, all positions of the point in spacetime will form a line of events. Because the motion is steady, this line will be straight. Because the point is moving towards the right the line will be tilted towards the right:



What would happen if the point were moving towards the left? What if it started somewhere besides the center? Try to see the appropriate diagrams before moving on.

The line swept out by our point is called the *world line* of the point. What would the world line of the point look like if it were moving slower or faster? It would be like this:



Summary:

It is natural, though strange at first, to think of time as a fourth dimension. It is not equivalent to the three spatial dimensions. One way this shows up is that we cannot stop moving through time.

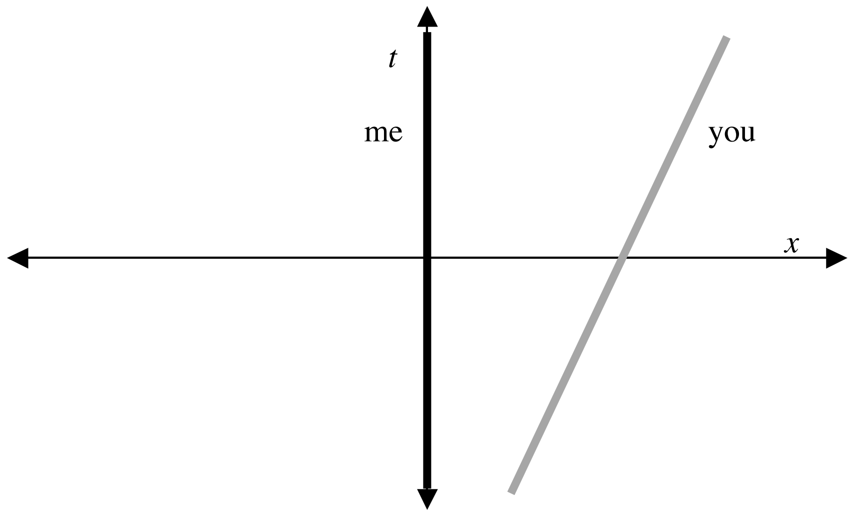
On a spacetime diagram, an object has one more dimension than we are used to. A point (zero dimensions) in three-space looks like a (one dimensional) line in spacetime. A rod (one dimension) in three-space looks like a (two-dimensional) area in spacetime.

A steadily moving point (in three-space) looks like a tilted straight line on a spacetime diagram. The faster the motion the greater the tilt.

Our First Foray Into Relativity: Time Dilation

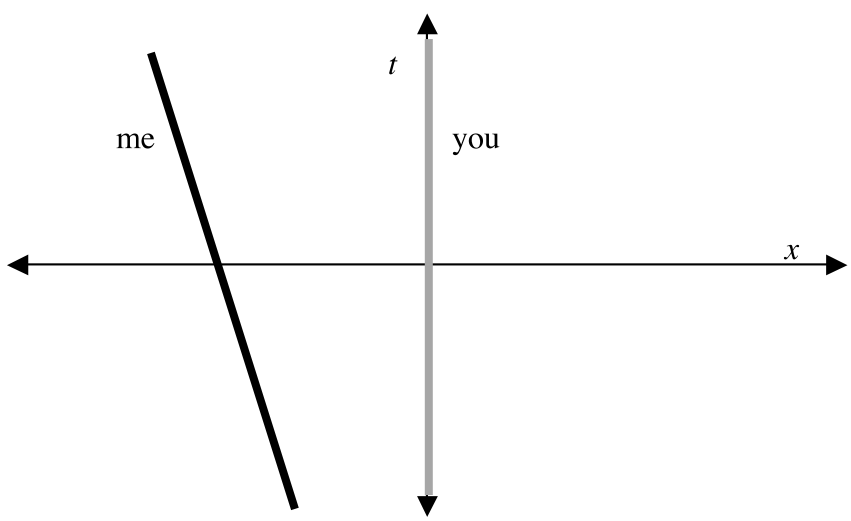
We are now going to look at what the statement "motion is relative" looks like in terms of spacetime diagrams. Let us imagine two spatial points. For the sake of brevity, I will call them me and you, taking advantage of the appropriate pronouns. Let's say that I am standing still, and you are steadily moving right:

What will this look like in a spacetime diagram? Since I am watching you move, I will put myself in the center of my diagram. As I am not moving in space, my world line will stay at x coordinate = 0, so my world line will coincide with my time axis. Your world line will be tilted towards the right:



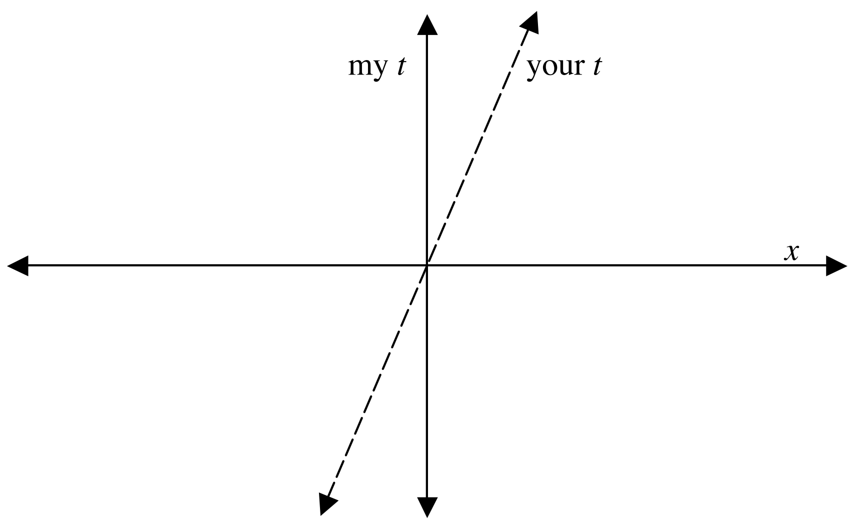
Now I decided that it is you who is moving. This is perfectly natural, since I am sitting watching you move along. Perfectly natural, that is, if you are me. However, you are not me (by definition), and so to you it would be natural to say that you are standing still and I am moving towards the left:

The situation has not changed! Physically, these two situations are the same. *We have only shifted our point of view!* What does the spacetime diagram look like for this point of view? First, it is only natural that you would put yourself at the center of the diagram. I would be to the left of you. Because I am, relative to you, moving steadily left, my world line will be a straight line tilted towards the left:



Notice something important about these diagrams. When we were making a spacetime diagram from my point of view, we had my world line coinciding with my time axis. This is a natural thing to do--it is the same as saying that I am always at my own position (my x coordinate is zero relative to myself). It also says that I measure my own time. After all, if I carry a watch with me, I am perfectly justified in using the ticks of this watch to measure my time. So taking my time axis to be along my world line is natural for several reasons.

Now from your point of view, for the same reasons given above (with you replacing me), your time axis will coincide with your world line. *But your time axis is tilted relative to mine--We do not have the same time axis and so we will not measure time the same!!!* In fact, I can ask what your time axis will look like from my point of view. In order to compare our world lines more easily, let's assume that we both are at the same place when $t = 0$. Then in my spacetime diagram your time axis is given by your world line



Our time axes do not (and cannot) coincide so long as you are moving relative to me. *This means that you and I will measure time differently!* This bears saying again.

Because you and I each carry our own time axes with us, and because when you move relative to me your time axis rotates relative to mine, we will measure time differently if we are moving relative to each other.

This is the insight behind so-called *time dilation*, or the fact that if you move relative to me we measure time differently. (Caution--It is not true that we measure the time of all things differently, only those that are in relative motion to you and in a different motion relative to me.)

At this point, we have only found that if you are moving relative to me we will measure time differently. We do not know enough to estimate the nature of this difference (slower, faster, whatever). Getting actual quantitative results will require more thinking. (In particular, you do not just try to draw equally spaced timing marks along both axes and somehow project. This will give an incorrect answer).

We know that your motion makes your time axis rotate from my point of view. We need to know what happens to the x-axis--Does it rotate by the same amount, not rotate at all, or something else? We answer this question by taking a closer look at a very unusual property of the speed of light.

The Speed of Light

Historically, Einstein's special theory of relativity was a theory about electromagnetism, and how space and time had to change relative to your state of motion in order for the equations of electromagnetism to be consistent. It was only later that Hermann Minkowski realized that the relativity of space and time showed that we live in a four-dimensional spacetime.

The relationship between electromagnetism and space and time becomes most clear when we consider the speed of light. The basic fact is that for the laws of electromagnetism to be the same for all observers,

the speed of light is the same for all observers.

In other words, if you are moving relative to me and we both observe the *same* light beam, then if we measure the speed of that light beam we will find the *same value*. This is not like the speed of normal objects: Say I am standing still on the side of the highway and you are going by at 30 miles per hour. We both observe a car that I measure to be going in the same direction as you at 60 miles per hour. You will measure that car to be going 30 miles per hour relative to you. This is natural and intuitive.

Light behaves completely differently. Say that I am standing on the side of the highway and you are now moving at half the speed of light. We both measure the speed of a light beam moving in the same direction as you. I measure that light beam to be moving at the speed of light, about 186,262 miles per second. You measure the speed of the same light beam relative to you to also be 186,262 miles per second, not the 186,262 / 2 = 93,131 miles per second that we would have expected.

How can this be? This is where special relativity tells us that when you move relative to me your space and time measurements differ from mine, and the specific way in which they differ is just right to keep the speed of light the same for both of us. I summarize this remarkable fact as

The Speed of Light is a Constant!!!

Motion as rotation of coordinates

We will answer the question about what happens to the x-axis in our spacetime diagram by using the fact that the speed of light is the same value for all observers. We will find that if the x-axis rotates backwards compared to what we would expect, then the speed of light will be the same from all points of view.

This strange behavior of the x-axis will bring us to seeing motion as a rotation of our coordinate axes that is different from rotations we are used to in three-space. Just as rotation in three-space preserves the distance between two points (this distance is the same no matter how you rotate in three-space), this new kind of rotation defines a new kind of distance that is preserved under these new rotations.

This new kind of distance will allow us to make quantitative estimates of how we observe space and time differently if we are in relative motion (though we will observe spacetime to be the same), and will give us the Lorentz transformation (the equations that tell us how our views of space and time are related).

We have discovered that time can naturally be thought as a fourth dimension, and that if you are moving relative to me then your time axis appears rotated relative to mine. This will mean that you and I will measure the time coordinates of an event differently. We now ask several questions:

What happened to your space (x) axis?

How does this all fit in with "The speed of light is a constant!!!"?

What are the quantitative implications of this rotation (exactly how do our measurements differ)?

Our path to an answer to these questions will be to define what happens to the x-axis so that the speed of light will be the same in all coordinates, rotated or not. This will force us to treat the rotation of coordinates due to motion differently from rotations in three-space. In order to do this we need to be more clear about how to represent the motion of light on our spacetime diagrams.

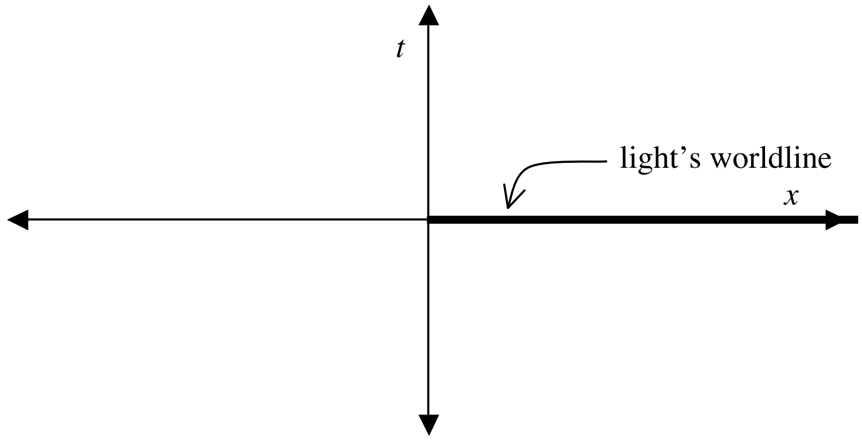
Putting light on our spacetime diagram

We can now ask "what does a beam of light look like on our spacetime diagram?" This is a subtle enough problem that it deserves it's own section.

This question already gets us in trouble because a light beam is a three-dimensional object. In order to put light on our spacetime diagram we need to think in terms of a flash of light that acts like a point in our one spatial (x) dimension. So for now, when I say 'light', I mean a point moving at the speed of light. Think this: That point on the x-axis lit up by the passage of a flash of light.

So let's put this moving point of light on our spacetime diagram. What does it look like? Like any point moving in a spatial direction, it looks like a tilted line on our spacetime diagram. Let's say it's moving from left to right and goes through the origin of our coordinates. Then the only thing we need to know is the tilt of the world line. This is given by it's velocity, and if we are measuring time in seconds and the spatial part in inches, then the velocity of light is about

11,802,852,680 inches/second. This is quite a tilt! Here's what it looks like on a spacetime diagram.



Light's world line is tilted so much that we cannot, on any diagram at an everyday scale, distinguish light's world line from the *x*-axis. Yet light's world line does not coincide with the *x*-axis, it is just very tilted.

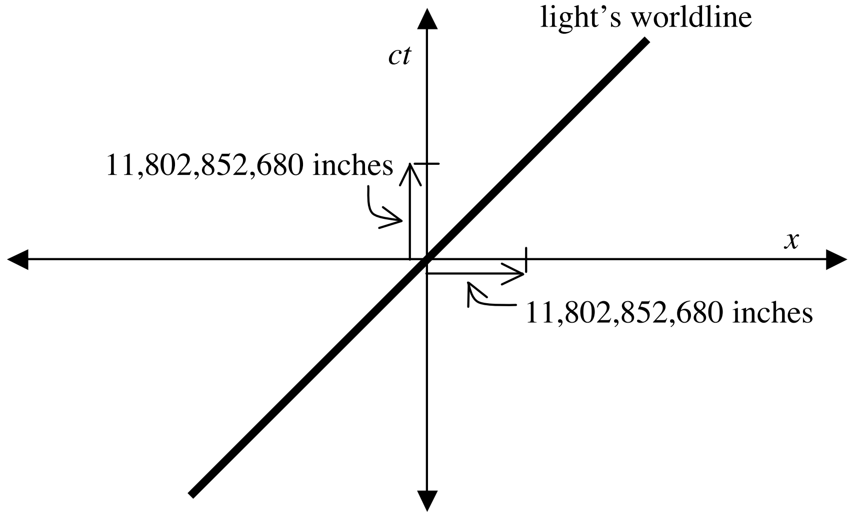
We need to think about the motion of light, and this way of measuring tilt does not help us. There is a way out. We can use our freedom of defining our units of measurement to make the tilt more reasonable. After all, we do not have to measure things with inches and seconds! We could use miles and hours, meters and minutes, or furlongs and fortnights. It turns out that no matter what units we choose, if we choose units that are in our everyday experience the speed of light (= tilt of light's world line) will be very large. This is simply because compared to things in our everyday experience light is very fast.

So if common units do not help us, let us make up our own. At the risk of being conservative, I wish to go on measuring spatial coordinate in terms of inches. I will now choose a new unit of time so that the tilt of light's world line is a reasonably visualizable one. I can choose this tilt, because I am making up my time unit. What tilt do I want? What can be more reasonable than a tilt of 1? I do this by simply taking my old time unit and multiplying it by the speed of light, measured in the old units. Lets say that my old units were inches and seconds:

New time unit=seconds multiplied by 11,802,852,680 inches/second.

What kind of unit is this? It is seconds multiplied by a whole lot of inches divided by seconds. The seconds cancel out, and the new time unit is 11,802,852,680 inches! We are now measuring both time and spatial directions in inches, though with a very different scale in the two directions. Let us adjust the scale on the *x*-axis, so that one *x* unit is 11,802,852,680 inches. We are still measuring spatial distances in inches, but now one mark is not one inch, but 11,802,852,680 of them.

What is the tilt of light's world line? Because light travels at 11,802,852,680 inches/second, it will travel 11,802,852,680 inches in one new time unit (I defined the new time unit so that this would be true). Thus in the new units, the speed of light is exactly 1, and so it's tilt is exactly 1:



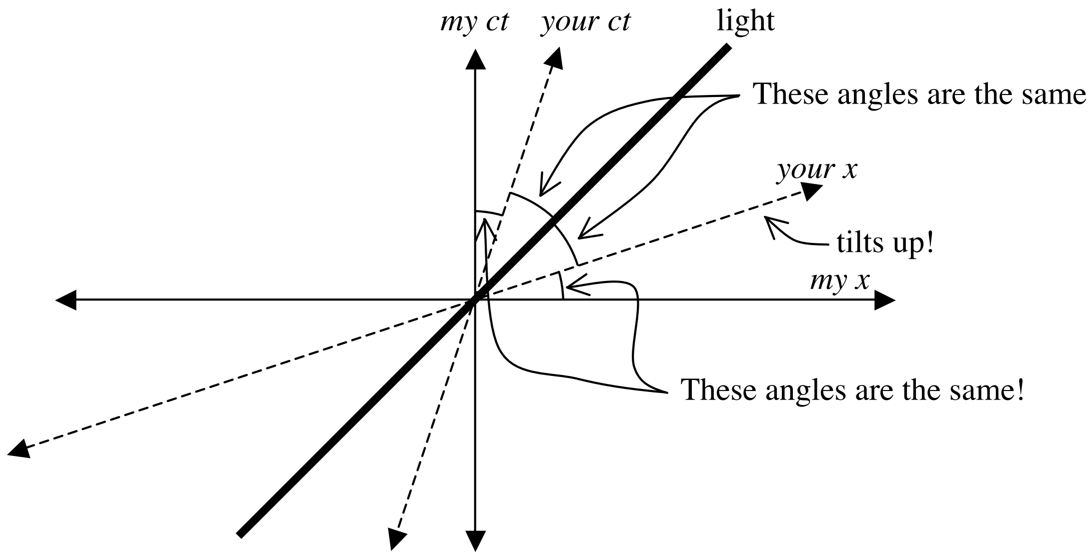
Let us give the new time unit a new label that will constantly remind us of how it is defined. It is defined as seconds multiplied by the speed of light. Now I am getting tired of writing 'the speed of light' and 11,802,852,680 inches/second again and again, so let me denote the speed of light by the letter *c*. Then my new time coordinate will be denoted by *ct*. In this way we will never forget how this new unit was defined.

All this was, admittedly, very dry and technical, but it has left us with a very easy way to visualize the motion of light on our spacetime diagram. With this new time unit, light's world line is tilted at 45 degrees. We now have a new way of saying "The speed of light is a constant!!!": "Light's world line has a tilt of 1 (is tilted at 45 degrees) in all coordinate systems (using the same units of measurement)!!!". This second phrasing will give us a precise way of finding it's implications.

The Lorentz transformation: The rotation of coordinate axes due to motion

We can now turn to the question of how your *x*-axis changes when you are moving relative to me. We will use the demand that the world line of light have a tilt of 1 in everyone's coordinate system.

Let us look more closely at the idea of tilt. Very picturesquely, the tilt of a line measures how close that line is to one axis compared to how close it is to the other axis. By how close, I mean the angle between the line and the axis. To say that a line has a tilt of 1 is to say that the angle that line makes with one coordinate axis is the same as the angle it makes with the other. Thus if, as your world line tilted in my spacetime diagram taking your time axis with it *your x-axis tilted towards the world line of the light by the same angle as your time axis tilted down*, then the world line of light would still make the same angle with each axis! The tilt of the light's world line would still be 1:



If your x-axis tilted towards the world line of the light the same amount as your time axis tilted down (also towards the world line of the light) due to your motion, then I would see that the world line of the light has a tilt of 1 in your coordinate system. In other words, in order for light to have the same speed for all moving observers ("The speed of light is a constant!!!"), the x-axis rotates backwards from what would be expected given the rotation of the time axis.

Now remember that these diagrams cannot be used to get the actual numerical values of the *your ct* and *your x* coordinates. To do this we must use actual formulas. Here are the formulas, where v is the velocity of the moving coordinates relative to mine, and c is, as usual, the speed of light:

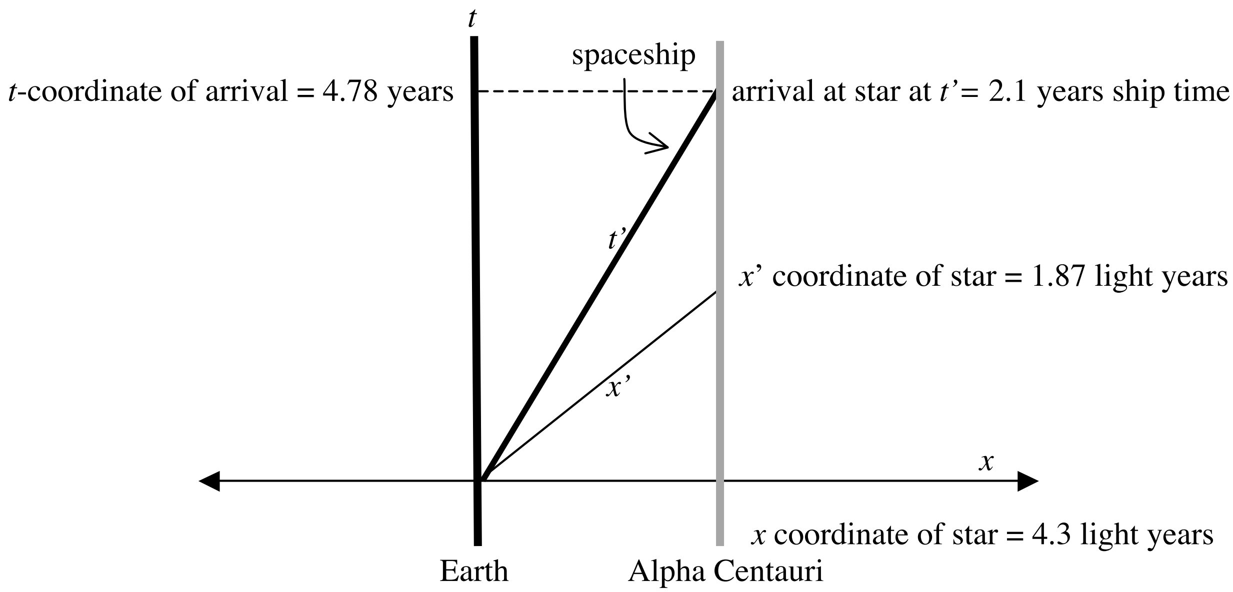
$$t = \frac{your\ t}{\sqrt{1 - \frac{v^2}{c^2}}},\ x = your\ x\sqrt{1 - \frac{v^2}{c^2}}$$

These formulas are true in all units of measurements, miles, seconds, etc. If you have a calculator with a square root function, you may wish to plug in some sample velocities and see what happens. You will discover that if v is much smaller than the speed of light, then $\frac{v^2}{c^2}$ is very small and so 1 minus this is still very close to 1. Because the square root of 1 is 1, the square root factors changing the coordinates in the above formulas are very close to 1 and so the coordinates are not very different. In fact, for speeds that we are used to, the difference is entirely undetectable. This is why we do not have an intuitive feel for relativity.

Enough talking--let's fly to the stars

Let's now see what this all means. We need to look at a case where you will go on a trip and travel at some speed close to the speed of light and go a very great distance. In this way, the difference in coordinates between us will be large enough to appreciate.

Say that you needed a real change of scene and decided to fly to the star Alpha Centauri, this being the nearest star. Even though this is the nearest star it is still quite far away, at a distance of 4.3 light years (=25,277,980,000,000 miles), and so you choose to travel at 90% of the speed of light, or at 167,654 miles per second (603,554,400 miles per hour). Let's say that you accelerated to this speed in an instant of time, not worrying about how you did this or whether or not you could survive this acceleration. Then this is what the situation would look like on my spacetime diagram (using *t'* and *x'* to denote your *t* and *x* axes):



We see that while you arrive at the star after 4.78 years have passed for me, the time that passed for you will be much shorter! In fact, the time that passes for you is only about 2.1 years. This is because at this speed the square root conversion factor in the above formulas is only about .436, and .436 times 4.78 years is about 2.1 years. Thus you would say that you arrive at Alpha Centauri after only 2.1 years of travel. If you immediately took off again and flew back to Earth at the same speed, only 4.2 years would have passed for you while for me 9.56 years would have passed. Thus I would have aged 5.36 years more than you since we last met! This is a very physical result of living in spacetime.

What if I managed to take a picture of your spaceship as you flew by at 90% of the speed of light? Let's say your ship is a mile long, so that if you orient your coordinate system so that the tail of the rocket is at your origin then the nose will have *x*' coordinate = 1 mile. For me, the *x* coordinate of the nose of your ship will be at 1 mile times .436 or at .436 miles. Your ship would appear shorter to me by more than half!