From last week:

In the last class, we went on a lightening tour of the local structure of spacetime. Part of that tour was slow and thorough, and part of the tour was very fast and cursory. The slow part covered the concepts that we will use in the rest of this course. The fast part talked about the rest of special relativity and will not be mentioned again. This latter material was mentioned for completeness only.

Here are the concepts from last time that are important to understand.

- *The world that we live in is in a four-dimensional space that we call spacetime*. This spacetime splits into three space dimensions (that we experience as ordinary space) and a time dimension. We learned that while we can move around in the three-dimensional space at will, in the time dimension we are constrained to move always in one direction at always the same 'speed'.
- *In spacetime, we always carry our own points of view*. That is to say we carry our own ways of measuring the time and position of events. When we observe the world we always do so from within our point of view.
- *In spacetime, objects that we perceive as points in our three dimensional space are lines pointed mostly in the time direction,* and objects that we perceive as things in three dimensional space (like you or me) are 'tubes' in spacetime. These lines and tubes in spacetime are called world lines.
- *We see the motion of other objects as the tilting of the object's world line*. Thus if you are moving relative to me, your world line is tilted relative to mine. Now your world line is part of what defines your point of view (in particular it coincides with your time direction), so your *point of view* is actually tilted relative to mine. Therefore when you move relative to me you will measure the time and position of events differently from me. This is the important fact to remember from the last class.
- *Your perceptions of the universe will be relative to your state of motion*. Is this to say that the universe is relative to your state of motion? Of course not! When considered as one unified entity, spacetime is the same no matter what your point of view. What happens when you move is that the way you split spacetime into space and time change. The spacetime remains the same.

We visualize spacetime by recognizing that because the three spatial dimensions are equivalent we can forget about two of them, leaving us with a two-dimensional spacetime made of one space direction and a time direction. This allows us to introduce spacetime diagrams, which were covered in detail in the handouts from last week. Each spacetime diagram is centered on our own point of view in spacetime.

It is the recognition that there is an underlying spacetime that is not relative to your point of view that is the core of the special theory of relativity, and was Einstein's driving insight when he developed the theory. We will now use this insight of a universe independent of our point of view in a new way. This will be by thinking about gravity.

Incorporating gravity

We get our first insights on how to incorporate gravity into the idea of spacetime by extending the idea of the relativity of points of view. We will say that because we experience gravity as an acceleration, the point of view of someone in a gravitational field is the same as someone who is accelerating (due to some other force). To understand this, we have to think a little bit more about our experience of gravity.

What is it like to experience the force of gravity? Very naively, there are two ways of experiencing gravity--falling and not falling. Close your eyes for a second and imagine first the experience of falling, then the experience of not falling. These two experiences of gravity are very different. In fact, when you are not falling you experience gravity as a force pulling you into whatever is under you. When you are falling you feel no force at all. In fact, if you fell in a closed room (such as an elevator whose cable had broken) you would have no experience of a gravitational force. It would be the (presumably disturbing) lack of feeling a force that would be remarkable to you.

So we have here two different experiences of gravity. I sit here typing, saying that gravity is what holds me to my chair, while you in your falling elevator say that there is no force at all. In fact, because you have no way to see that you are moving you will think (and perhaps hope) that there is no gravity! Yet your point of view is just as valid a way of observing the universe as mine. So are we to say that the force of gravity is not an objective thing--that it depends on our state of motion? This line of reasoning disturbed Einstein greatly and led him on a long (1907-1913) and somewhat confused path to General Relativity. We will not follow his path, but use his starting point combined with hindsight to jump to Einstein's final conclusion.

The principle of equivalence

We are going to go on the presumption that gravity is a very physical thing and is not relative to our state of motion. So how do we account for the tremendously different observations that we can get by looking at gravity from different states of motion? We will, following Einstein, make the following assertion:

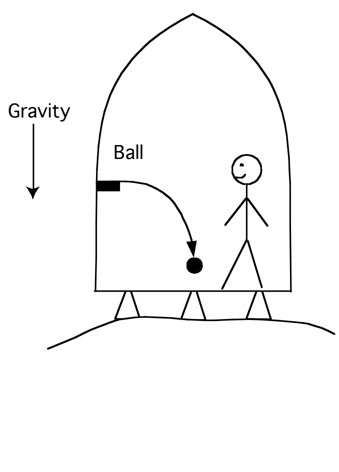
Someone who is experiencing a gravitational force is taking the same point of view as someone who is accelerating.

This says that there is no way to tell if you are experiencing gravity or if you are somehow being accelerated. In both cases the experience is the same. This is called the *principle of equivalence*.

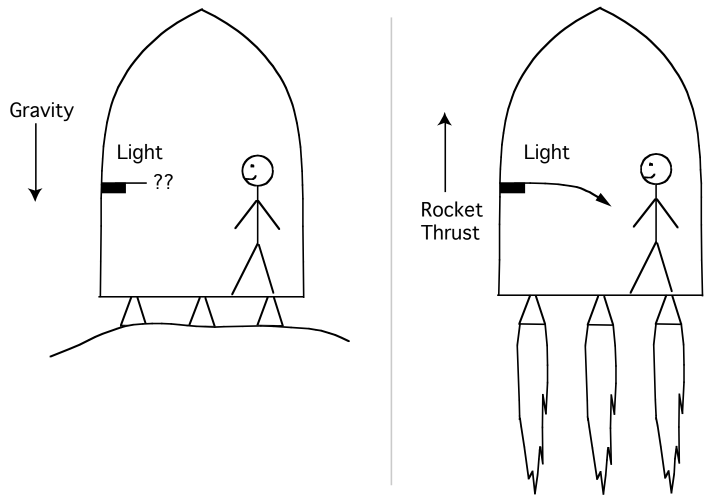
First, let us make sure that this principle fits our experience. Why are gravity and acceleration so related? Primarily because they feel the same. Perhaps the strongest acceleration any of us have felt is in an airplane. As the plane speeds up you feel a force pushing you into your seat. This feels like just as real a force as gravity, but the acceleration you feel is generated by a change in your state of motion. Thus acceleration feels like a force even though it is distinct from the gravitational force.

The principle of equivalence says more than "gravitational forces and accelerations feel the same". It says that the point of view that you have in one is the same as the point of view you have in the other. A more traditional way to say this is that there is no way for an observer in a closed box to tell if he is experiencing the force of gravity or if he is accelerating (for this to be strictly true the box has to be very small). Let us take the role of that observer.

Imagine your self in a box sitting on the Earth. You feel your weight. If you drop something you see it fall and if you watch it carefully enough you will see how it accelerates. Now imagine yourself in a box out in space, away from any large bodies so that there is no gravity, with the box being towed by a rocket that is accelerating. You feel yourself pressed to the floor, and it feels like you have weight. If you drop something you will see it move towards the floor of the box, because when you let go of it it stopped accelerating. If the acceleration of the rocket had the correct value (0 to 60 mph in about three and three quarter seconds or 9.8 meters per second squared) then your weight and how fast your objects fall would be the same as they would be in the case in which you were sitting on the Earth. There would be no way for you to tell which case it was without looking out the window.



The principle of equivalence: A moving ball viewed from the inside of a rocket. Left: the rocket sitting on the ground under the influence of gravity. Right: the rocket in space (far from any other mass) accelerating with the same acceleration as Earth's gravity. On the right the ball's path is curved from the point of view of the rocket, but the ball's path is actually straight from a non-accelerated point of view.



An application of the principle of equivalence to figure out what light does in gravity. On the left, a naïve application of Newton's formulas for gravity does not work because light has no mass. But on the right the light's path will be curved from the point of view of the astronaut (because the astronaut is accelerating – the light path is actually straight from a non-accelerating perspective). Therefore according to the principle of equivalence light's path will be curved by gravity.

The transition to curvature

We are now in a position to think about how spacetime fits in with gravity. If someone experiencing gravity is taking the same point of view as someone who is accelerating, then all we have to do is look at the spacetime point of view of someone who is accelerating.

Acceleration is the changing of your state of motion (here the word 'accelerating' refers to speeding up, slowing down, and turning). Now when you change your state of motion, you are tilting your way of separating spacetime into space and time. The faster you go relative to me the more tilted your point of view becomes. Therefore as you accelerate you will experience the space and time of the universe curving! As you go faster and faster you will see lengths in the universe shorten and the time things take lengthen.

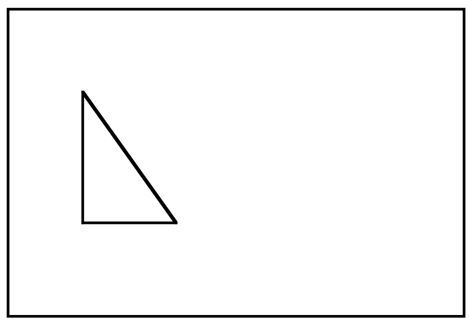
Now if, according to the principle of equivalence, when you experience gravity you are taking the same point of view as someone who is accelerating, then you must experience something like this curving of the space and time around you. This is, admittedly, very hard to imagine, and in fact I feel that this chain of logical reasoning is quite shaky. It does, however, introduce us to the idea that in some way gravity curves space.

This line of reasoning is not the one used by Einstein. Einstein used the principle of equivalence in combination with the demand of the constancy of the speed of light to make the first of several theories of gravity. Qualitatively, this theory that Einstein created (in about 1908) made qualitatively correct predictions about the effect of gravity on light, but these predictions were numerically only half of what was later observed. Einstein did not know this, however, and he was dissatisfied with the reasoning that led to these theories. It was this dissatisfaction that took Einstein on the path to the General Theory of relativity.

How to Describe Curvature

What is a curved surface? We know how to picture two dimensional surfaces that are curved into the third dimension...Just look around you to see some. So at least for two dimensional surfaces we have a good intuitive ability to describe curvature. But we want to talk about the curvature of spacetime, which is four dimensional! How can we visualize a curved four dimensional space? This would require us to visualize a fifth dimension for the space to curve into, which we certainly cannot do. Thus trying to directly visualize the curvature of spacetime is a hopeless task. So how else can we describe this curvature?

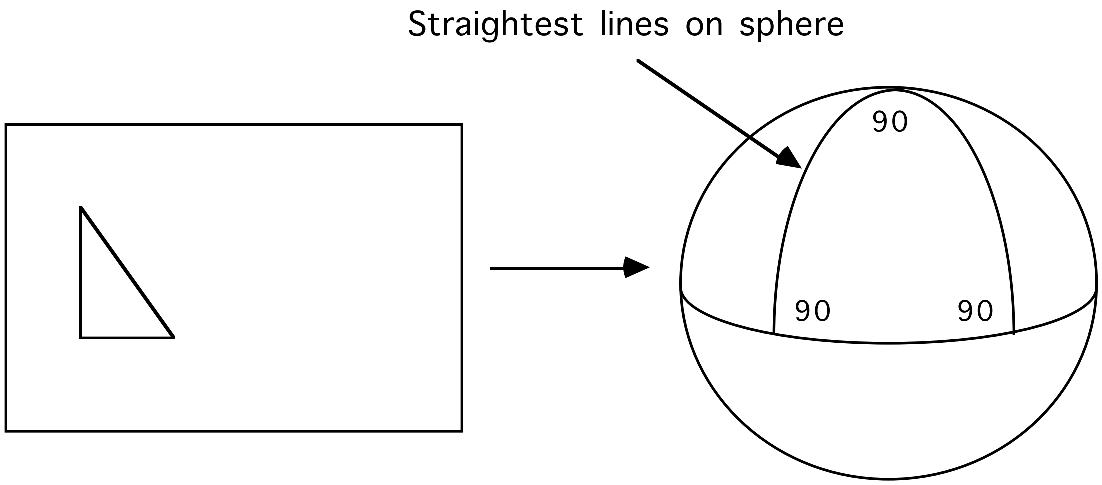
It turns out that we do not need to visualize curvature in order to describe it mathematically. Consider a triangle on a flat plane. You may remember that if you add up all the interior angles (in degrees) of the triangle you will get 180 degrees. This is true on a flat plane no matter what triangle you draw so long as the sides of the triangle are straight lines:



Total of interior
angles of triangle = 180

Now consider this triangle drawn on a sphere (= surface of a ball). Here we run into our first problem: On a curved surface you cannot draw a straight line that stays on that surface. Yet rather than think about straight lines as simply straight, we can think of them as lines that never turn. On a flat surface this is exactly what we mean by straight, but we can generalize this definition to curved surfaces. Simply pick a point and a direction on a curved surface, start moving in that direction staying on the curved surface and not turning (always go forward). You will then trace out the straightest line in that direction from that point on that curved surface. On a sphere, the line that you trace out is called a great circle and is the shortest line on the sphere between the starting point and the ending point of that line. This is why airplanes fly in great circles.

Now let's try again to draw the triangle on the sphere. Imagine a triangle on a sphere whose sides are great circles (really try to picture this). We may, if we wish, put one corner of the triangle at the north pole and the other two on the equator. Then we may choose the internal angle of the corner at the north pole to be 90 degrees. Because the other corners are at the equator their interior angles will necessarily also be 90 degrees:



Total of interior angles of triangle = 180

Total of interior angles of triangle = 270

Note that the interior angles of the triangle on the sphere add up to 270 degrees, which is greater than 180 degrees! If the triangle were smaller, we would have the interior angles adding up to some other number, but because the triangle is on the sphere this number will always be greater than 180 degrees. In fact there is a general rule:

On a surface of positive curvature the interior angles of a triangle will always add up to a value greater than 180 degrees.

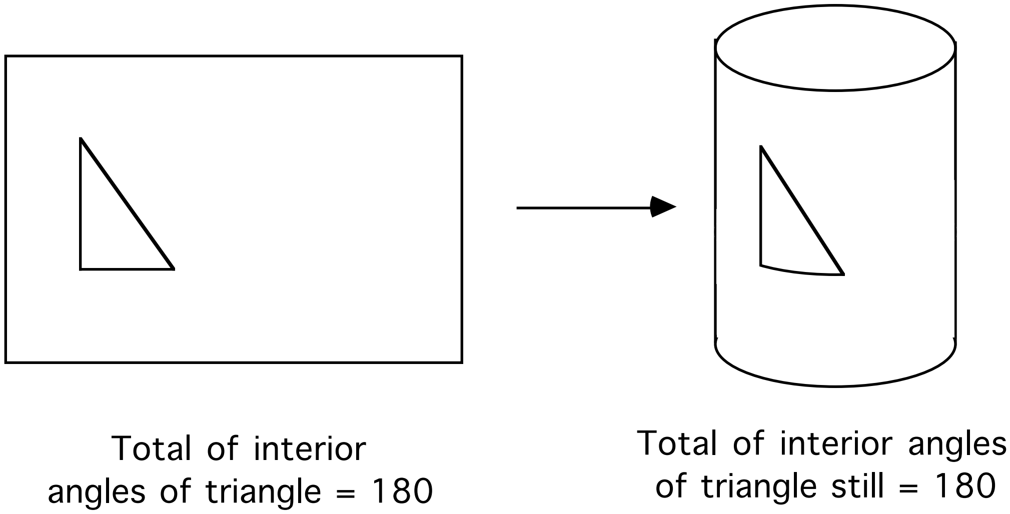
There are also surfaces that would have triangles that would add up to less than 180 degrees. We say that the curvature of these surfaces is negative. There is the corresponding rule:

On a surface of negative curvature the interior angles of a triangle will always add up to a value less than 180 degrees.

We could, for example, determine the curvature of the Earth by measuring the interior angles of a sufficiently large triangle (this would be difficult to do with the required accuracy in practice). This method uses triangles in two dimensions, but we could generalize it to multidimensional objects in higher dimensional spaces.

Note that we are measuring the curvature of a space without having to, in any way, stand outside of that space. This means that the higher dimensional space that we need to visualize curvature may not and usually does not have any physical meaning or existence. This fact is something that we will have to keep very much in mind in this course.

Note that this method of measuring the interior angles of triangles will tell you that many spaces that you would have said are curved are flat. For example, we could take our flat plane with a triangle and roll it into a cylinder without distorting the angles of the triangle:

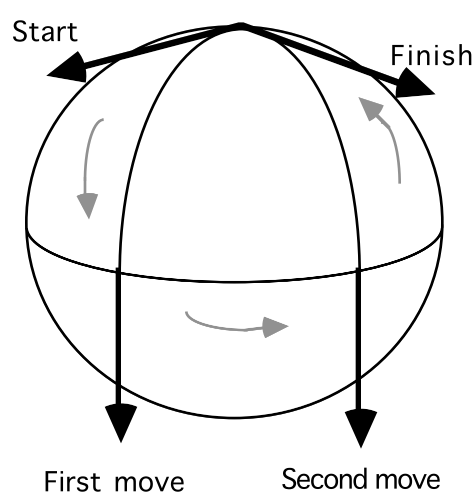


(Why can't you do this with a sphere?)

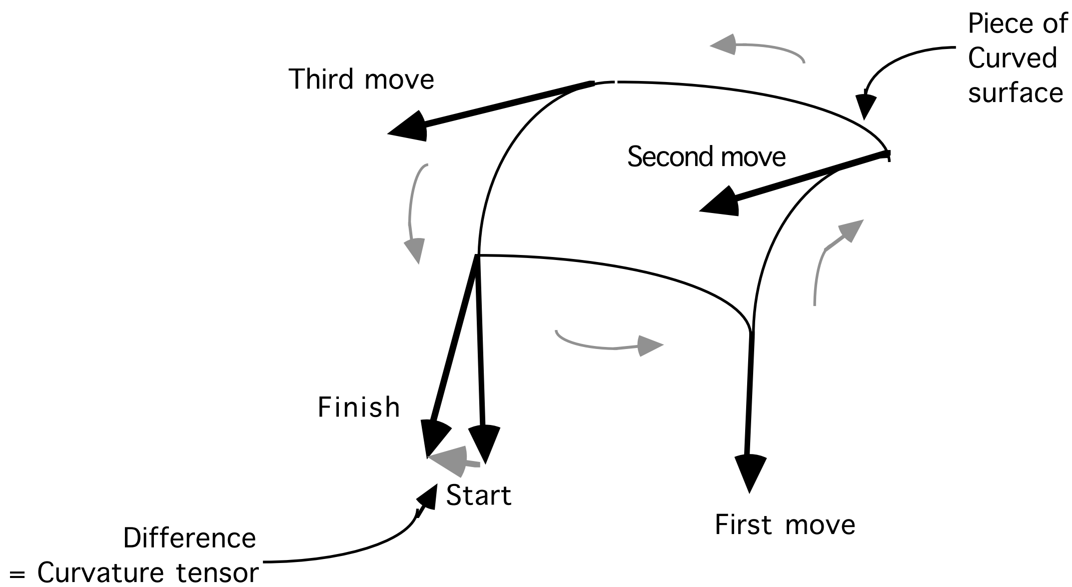
Thus a cylinder is flat for this method of measuring curvature. This actually matches what we want as a description of curvature, for as we shall see later the cylinder would have the same physics as the flat plane (in particular no gravity).

This method of measuring interior angles of triangles is a good way of measuring curvature, but it is actually rather clumsy to use in an abstract mathematical way. There is another way that is easier to abstract and generalizes very easily to higher dimensions.

Consider the sphere again. Imagine an arrow at the north pole of the sphere that is tangent to the sphere. Now imagine moving the arrow along the sides of the triangle in the above example so that it remains always tangent to the sphere but otherwise does not turn:



When it gets back to the north pole, it will have been rotated from its original starting position! The value through which it was rotated (relative to the area of the triangle) is a measure of the curvature of the sphere. This is very close to how mathematicians define what is called the Riemann Curvature Tensor (denoted *R*). *R* completely describes the curvature of any space. It is defined as the rotation that a vector goes through as it circumnavigates an infinitesimally small square on the curved surface:



The significance of the curving of spacetime is in how our world lines, which are the straightest lines in the curved spacetime, are curved by the spacetime. We will see what the significance of this is next week.