Relating Curvature to the Real World

Recall the mantra for the class:

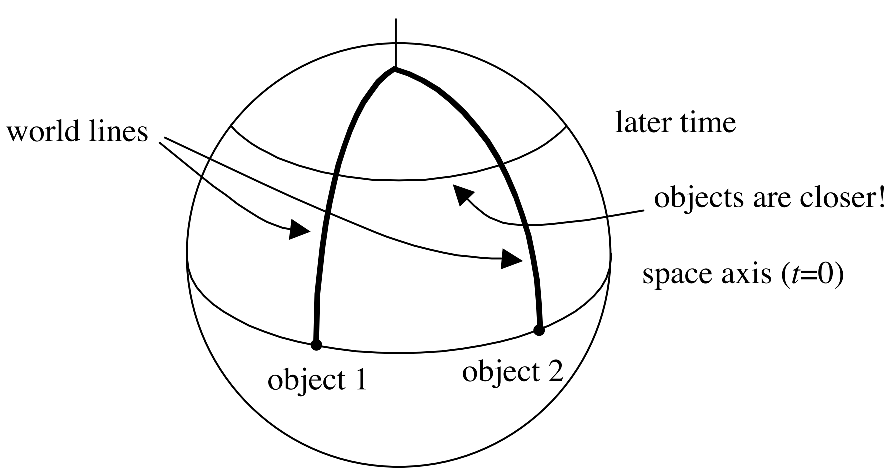
"*Space acts on matter, telling it how to move.*
In turn, matter reacts back on space telling it how to curve"
-- John A. Wheeler

The Physical Effects of Curvature

As I have said many times in this class, we cannot directly visualize the curvature of the space we are in. There is another way of saying this--we cannot make any measurements of the curvature of spacetime that in any way feels like a measurement of curvature. Now of course by feeling gravity we are measuring the curvature of spacetime all the time, but it does not look like curvature. The physicist has the same problem. What physical quantities that are related to curvature are the most convenient for the physicist to measure?

The most physical effect of curvature is to observe the paths of objects as they move in the straightest possible line in a curved *spacetime*. Recall the spacetime diagrams from the second class. The straightest lines on those diagrams were actual straight lines, because in the second class we were talking about a *flat* spacetime. Let's imagine a little toy example which will help us imagine curved spactime. This is not a physical example, but it will introduce the concepts we need.

Imagine a spacetime diagram on a sphere, with the space axis on the equator of the sphere. Consider two objects, not moving relative to each other and starting at two different locations on the space axis (on the equator, at the "now" moment of time). The world lines of these objects will, of course, point in the time direction of this spacetime diagram on a sphere, which is pointing towards the north pole. Because, in this example, our spacetime is two dimensional and is on the surface of the sphere, the world lines of the objects will also stay on the surface of the sphere. But because the objects are not moving their world lines always point in the time direction (of this spacetime diagram), which is towards the north pole. Therefore the world lines will be great circles from the equator to the north pole. At later times the objects will be closer together, so it will appear to each object that there is a force pulling the objects together. This is exactly like the ants on the apple story of the first class, but with a simpler geometry.



From this example we can see where the "motion" in the phrase "moving along the straightest line" comes from: it is the motion forward in the time direction of spacetime that we always experience and cannot stop.

(You can imagine doing this yourself in space (not spacetime) if you imagine you and a friend walking to the north pole of the Earth. No matter how far apart you start, so long as you start at the same latitude, you will eventually find yourselves moving closer and closer together, even though you are both walking towards the north pole. Remember: it doesn't matter how heavy you are or how you are traveling, you will both follow the same path from the same starting point (like gravity) and you will move closer together even though you don't feel a force pulling you together (like gravity)!)

The above example was of a two-dimensional spacetime on the surface of a sphere. Real spacetime is four-dimensional, so the actual curvature is typically much more complex. In flat spacetime we used the fact that all the space directions are the same (because spacetime in this case was flat in all directions) to reduce our picture to a two-dimensional spacetime diagram. We cannot typically do this when spacetime is curved, however, because it can (and usually is) curved differently in different directions.

Therefore to visualize spacetime we usually look at the straightest lines, or *geodesics*, and watch how things that move along geodesics change their distances from each other. After all, if two objects are moving closer to or further from each other, then they are moving relative to each other. If the objects started out with no relative motion (like the objects at the equator in the sphere spacetime example above) and then after time passes the objects are closer together, then it will seem that they developed a relative motion over time. In other words, from the point of view of the objects, they were accelerated. This is why curved spacetime looks to us like it causes the acceleration of objects, which is the primary physical effect of gravity.

This is how "space acts on matter, telling it how to move".

How to Determine the Curvature

The second part of our mantra is "matter reacts back on space telling it how to curve". Physicists determine the curvature of some region of spacetime by first looking at how matter is distributed in that region. But before we can get some idea of how physicists determine curvature we should say a little bit about how they describe curvature.

There are many ways to perceive and measure curvature. The ways you would intuitively measure curvature requires you to stand "outside" and say how much the surface is curved. But we cannot stand outside of spacetime, so we need a way to describe curvature inside spacetime. Physicists use a description that is closely related to the idea of geodesics, which is the idea of the "shortest distance between points". The technical description of the shortest distance between points is called the *metric*. You are familiar with the metric formula in flat space: it is just the Pythagorean theorem (that for a right triangle, the square of the hypotenuse is the sum of the squares of the two sides).

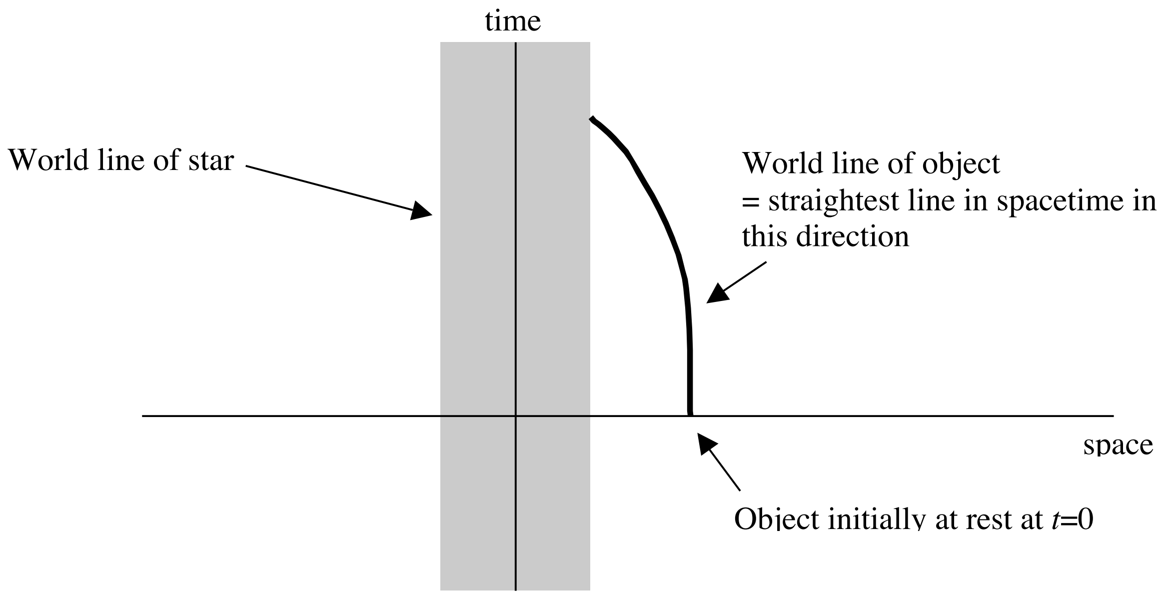
Let's compare the shortest distance between two points on a plane and two points on a sphere. On a plane the shortest distance is, of course, the length of the straight line between the two points. The shortest distance between two points on a sphere is the length of the great circle connecting the two points. Now think for a moment about the *three*-dimensional space containing both the plane and the sphere. Then the distance in three-dimensional space between the points on the sphere is given by the length of the straight line between the points (in 3D). Similarly the distance in three-dimensional space between the points on the plane is given by the length of the straight line between the points, which is the same line in 2D and 3D. Let's assume that the 3D distance between the points is the same for both the sphere and the plane. Now here is the important observation: notice that the distance between the points *measured on the sphere* is greater than the distance between the points on the plane. Therefore we can use the distance between points to measure curvature.

The formula for the distance between points in a curved space is called the *metric*. The curvature tensor (called R in the notes from the first session) is completely determined by the metric. Also the formula for a geodesic is expressed entirely in terms of the metric. When a physicist solves for the curvature given a distribution of matter, what is really happening is the solution is for the metric, when the formula for R in terms of the metric is substituted into Einstein's equations. We then try to find a metric that satisfies that equation for the given distribution of matter. The resulting formulas are very complicated and difficult to solve.

Actually, the equations of gravitation are never (to my knowledge) actually solved this way. It is simply too difficult a problem to try to exactly match a specified distribution of matter with the correct metric. What physicists actually do is make simplifying assumptions about the distribution of matter. In particular they may specify that the matter takes, for example, the form of a rotating sphere that does not change with time (this particular assumption leads you to what is known as the Kerr solution). This kind of the assumption greatly limits the possible mathematical forms that the distance element may take and so makes the finding of the correct distance element easier.

Let me describe a real life example. This is the solution for a distribution of matter that is a non-rotating, non-changing sphere. In this case, the matter looks the same from every direction. Because the sphere is the same from every direction, only the distance from the sphere matters. It turns out that this implies that the space around this sphere is only curved in the spatial direction pointed towards the center of the sphere and in the time direction. Therefore at a point in spacetime near the sphere one can choose your coordinate axes so that there is no curvature in two of the spatial directions. This makes it much easier to solve for the metric in this case.

This is a very useful example because it is a good approximate model for the distribution of matter in a star. Stars are approximately spherical, and they rotate and change shape very slowly. Therefore the metric in this simple example will describe the space around a star. This is called the *Schwarzschild solution*. Physicists use the Schwarzschild metric to calculate the straightest lines in the space around the sun. Sure enough, the straightest line followed by an object initially at rest relative to the sphere will follow a parabolic path (in spacetime) towards the center of the sphere.



Spacetime diagram of curved spacetime around a star in the time – space directions, where the space direction is the direction towards the center of the star.

Also, the straightest line pointed in certain directions in the curved space around the star form almost closed ellipses! This is the path followed by the Earth in its orbit around the sun. Again, the Earth is following the closest thing there is to a straight line in the space curved by the sun. Therefore the Earth in it's orbit is testing General relativity all of the time!

Of course Newton's theory of gravity also predicts that the Earth will orbit the sun in an ellipse. It is important to find examples where Einstein's theory makes predictions that are measurably different from Newton's theory. There are other tests of general relativity that are more interesting because they predict things that were not previously understood. One of them is the motion of the planet Mercury around the sun. I mentioned above that the straight lines that form orbits are almost closed ellipses. This is simply how it works out in this curved space. According to the Newtonian theory of gravity, orbits form really closed ellipses. It was observed for many years before the advent of general relativity that Mercury's orbit does not form a closed ellipse in a way that could not be explained by the gravitational influence of the other planets (according to Newtonian gravity). General relativity, however, exactly predicts the observed lack of closure in Mercury's orbit.

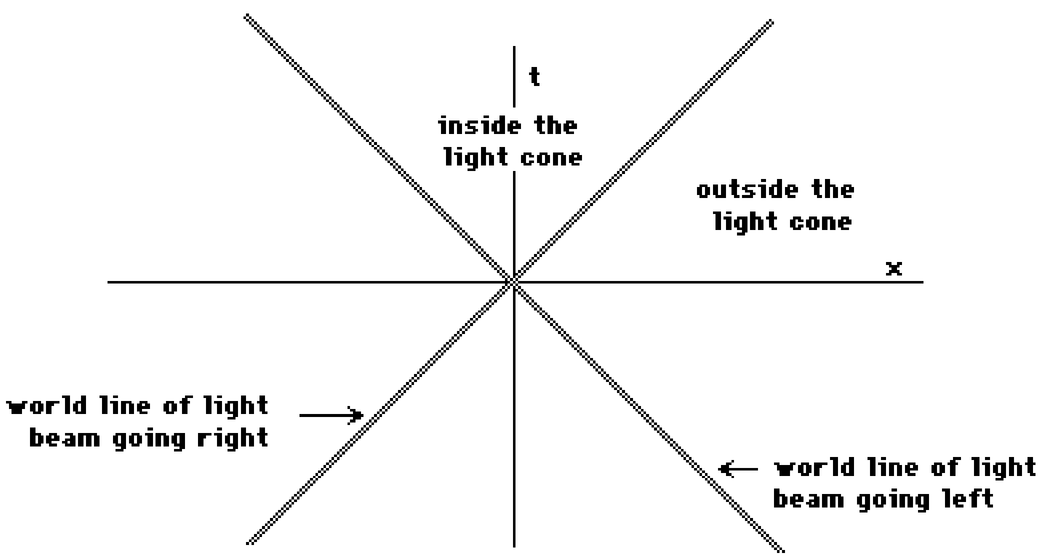
The other two classical tests of general relativity involve how light is effected by gravity. According to Newtonian gravity, if light is a particle with an effective mass generated by the energy of the light (remember that according to special relativity energy has an equivalent amount of mass (E=mc2)) then gravity should effect light in a certain way. According to general relativity light (like everything else) simply moves along straight lines in a curved space and so will be effected by gravity in a way that disagrees with Newtonian gravity. When you measure the effects, general relativity gives the correct prediction.

When the sphere of matter is rotating, the curvature of the spacetime around the sphere is a little more complicated. In this case there is a curvature in the direction of the rotation, which has the effect that geodesics of objects that are initially at rest with respect to the sphere will be pulled slightly in the direction of rotation, rather than falling directly towards the center of the sphere. This effect is very small and very hard to measure for the Earth's spacetime curvature. Stanford's gravity probe B experiment will attempt to measure this effect with a satellite in orbit. Because the effect is so small the gravity probe B experiment will try to accumulate the effect over time.

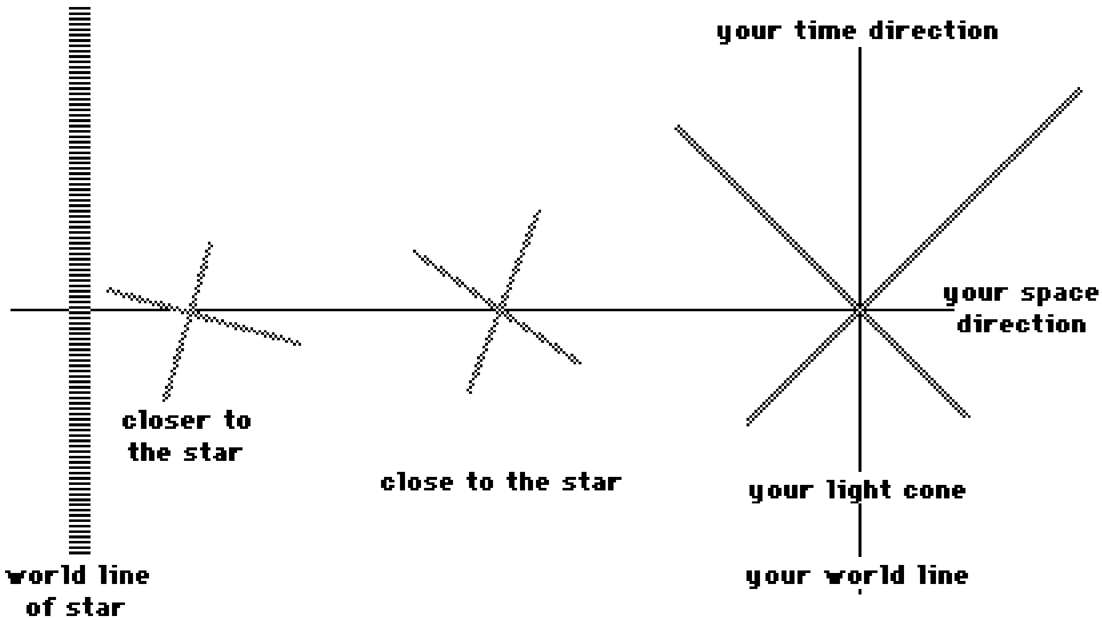
Visualizing the Schwarzschild Geometry: Black Holes

Recall the spacetime diagrams from when we were talking about special relativity. Light plays a very special role in spacetime, in that the speed of light is always the same no matter what your point of view. In particular, even if you are moving relative to me we will both measure a light beam to be travelling with the same speed.

Put in spacetime diagram terms, this means that the tilt of light's world line will be the same no matter what point of view in spacetime we take. Like in the second session, we can manipulate our units of measurement so that on our diagram this tilt is that of a line at 45 degrees. Let us agree to do this, so that our spacetime diagram looks like this:

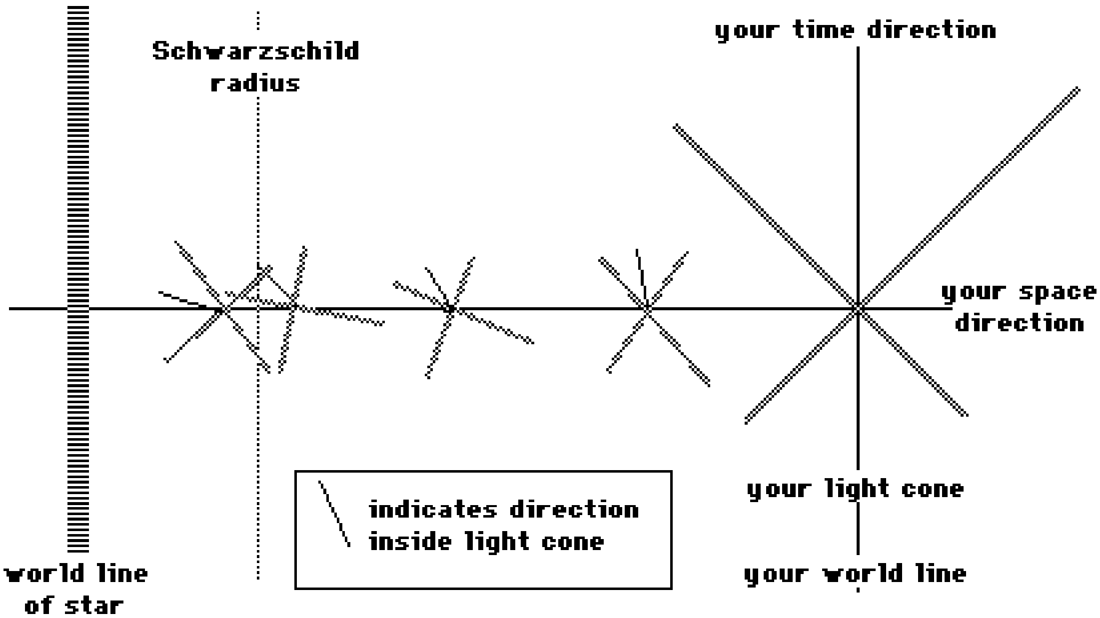


We call the two lines representing the world lines of light beams going opposite directions the light cone. The light cone from the center will always look the same in any point of view. If, however, you were to look at the light cone at a point away from your center in a curved space then that light cone may be tilted. Here is a drawing of the light cone structure of the spacetime around a star according to the Schwarzschild solution:



The effect of the curvature on the light cone is to cause the entire light cone to tilt (from the point of view that we take far away from the star) towards the star. This helps us to visualize the curvature around the star.

It is only the surface of the star that stops the light cone from turning over completely. One can therefore ask what would happen if the star became smaller and smaller. Then the space close to the surface of the star would become more and more curved and the light cones would tilt more and more over. When the star reaches a small enough radius (3 kilometers in the case of the sun) the light cone at it's surface has tilted completely over (remember that this is from our point of view far away from the star). This special radius is called the Schwarzschild radius. Here it is visually:



If the star is smaller than it's Schwarzschild radius, then the space at the surface is so curved over that the entire interior of the light cone is pointed at the star! Now as nothing can ever travel to a point in spacetime outside of it's own light cone, this means that any object starting from a point inside the Schwarzschild radius will find the surface of the star in it's future no matter how it moves! Another way of saying this is that for someone inside the Schwarzschild radius of a star all paths move towards the star. Even to stay at the same distance from the star would require our unfortunate friend to travel outside his own light cone. This would require him to travel faster than light, which you cannot do due to the structure of spacetime.

So anything that falls closer to the center of a star than it's Schwarzschild radius will necessarily eventually hit the star--including light. Any object that becomes smaller than it's own Schwarzschild radius is called a black hole. It need not have started as a star, but stars are the only objects in nature that we know of that can shrink to a size small enough to become a black hole.

I should mention that there is no Schwarzschild radius inside a star, for the matter inside the star keeps the curvature there well behaved.

What is the experience of passing through the Schwarzschild radius of a star? It is nothing special at all. For the person falling into the star, there is no Schwarzschild radius. This is because this person always carries his own point of view and in his point of view his light cone always points up. The Schwarzschild radius is only a feature of the comparison between the point of view that we have far from the star and the point of view near the star. This is not just a simple comparison, for it has very physical effects. One effect of very large curvature near a star is that time runs slower near a star than farther away. Thus if you traveled very near (but not through!) the Schwarzschild radius of a black hole and returned I would have aged much more than you would have. In fact, if I watched you approach the black hole, I would see your time slow down more and more as you got closer and closer to the Schwarzschild radius. You, on the other hand, would think that nothing particularly strange was happening to yourself, but you would see the rest of the universe speeding up.

What if you let yourself fall through? As I mentioned before, you would fall through the Schwarzschild radius and very quickly (much less than one second after you passed through the Schwarzschild radius) hit whatever it is in the center. I, however, am sitting at the safety of a great distance and would see you slowing down more and more as you approached the Schwarzschild radius. In fact, I would never see you pass through the Schwarzschild radius. Due to the curvature of the space made by the star it would take an infinite amount of my time for you to reach the Schwarzschild radius. In my way of measuring time you would never get there. Yet in your way of measuring time you get there rather quickly. Thus we live in a universe where something can happen to someone which never happens at all as far as someone else is concerned. I consider it a good thing that you could never come out of the black hole again and confront me with an event that, as far as I am concerned, has not yet happened.

Once something is inside the Schwarzschild radius of a black hole, it can never get out. This means that someone inside the Schwarzschild radius can never send anything out to the outside universe. Thus we, sitting off in the distance, can never receive any message from within the Schwarzschild radius. In particular, we can never receive any information about the black hole itself. This is why John Wheeler coined the term 'black hole' for this kind of object. This radius is also known as the event horizon. As we can not receive any messages from within the black hole, we cannot know the particular structure of the hole. The only things that we can know about the objects inside the Schwarzschild radius are the mass, electromagnetic charge, and total amount of rotation in the matter, as all of these things contribute to the spacetime curvature around the object. Further, as the world lines of all objects inside the Schwarzschild radius eventually hit the surface of the star, one can show that the matter inside the black hole must take a spherical shape. This is called the "Black holes have no hair theorem" and was first proved by Stephen Hawking and Roger Penrose.

Hawking and Penrose were the main researchers who studied the overall properties of the curvature of spacetime, introducing very pretty and abstract mathematical geometric techniques. They proved several so-called singularity theorems, which say that under very general and physically likely circumstances spacetime will curve in such a way that all of the dimensions of spacetime will be squished down to a point. Such a point is called a singularity. Some of the singularity theorems refer specifically to black holes and imply that once a star has become smaller than it's own Schwarzschild radius, then it must inevitably collapse to a point taking it's surrounding spacetime with it. This is something that physicists do not like, as it implies that there are points in spacetime where the structure of space somehow breaks down. It is generally assumed, but by no means known, that when the black hole becomes small enough general relativity breaks down and is no longer true due to quantum mechanical effects. Thus as the black hole becomes small enough general relativity no longer applies and as no one knows what does apply all bets are off.

It would be very interesting, therefore, to try to observe what happens to a collapsing star inside it's own Schwarzschild radius. Now we simply cannot do this because of the nature of the Schwarzschild curvature. Other solutions of Einstein's equations, though, do offer the possibility of creating spacetime singularities not enclosed by an event horizon (the event horizon is the generalization of the Schwarzschild radius to these other solutions). If we are looking at a sphere of matter that is electrically charged or spinning (or both), then the spacetime curvature around the star is somewhat more complicated (though not so complicated as to prevent figuring out these solutions). These solutions are called the Reisner-Nordstrom solution and the Kerr solution respectively. Here if the charge or spin is not too large there is an event horizon surrounding the black hole, but if the charge or spin are large enough then the events horizons vanish and you have a singularity of spacetime for your examination.

So if we try to create a highly charged black hole, say by dumping in lots of protons to give the hole a large positive charge, perhaps we can get a glimpse of the singularity. The trouble is that as the hole becomes more and more charged it becomes harder and harder to dump the protons in due to the electrostatic repulsion of like charges (both the hole and the protons are positively charged). It turns out that long before the hole would be charged enough to make the event horizon vanish it would be effectively impossible to keep dumping protons in. No one has been able to invent even a theoretical way to keep dumping protons in.

Similarly, for a spinning black hole the spacetime curvature is such that as objects approach the black hole they start spinning with the hole (this is called dragging of inertial frames). Say we try to make the hole spin faster by throwing things in in such a way as to add to the spin of the total system. As the hole spins faster and faster approaching objects get dragged faster and faster. Long before the spin of the hole is fast enough to eliminate the event horizon any object approaching the hole would be flung back out into space by the dragging before it could fall in.

While the above analysis depends on estimates of what we can actually do, the fact that they do not seem to come even close to canceling the event horizons has led physicists to postulate what is called 'the cosmic censorship hypothesis': There are no singularities in the universe that are not surrounded by event horizons. In other words, there are no naked singularities. At this point in time, despite large effort by very good physicists, this is still a hypothesis. No one knows if it is true or false and it remains one of the most important outstanding problems in general relativity.